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Aggrandization of Spaces of Holomorphic Functions Reduces to Aggrandization on the Boundary*

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Abstract—We show that grand spaces of holomorphic functions may be equivalently defined in terms of aggrandization related only to the boundary. We base ourselves on recent studies of the so-called local aggrandization of Lebesgue spaces and extent this approach to the case of arbitrary Banach spaces of functions on metric spaces. We apply this approach to prove, in the case of Bergman and Bergman–Morrey spaces on the unit disk, that these grand spaces may be equivalently defined as grand spaces with weighted aggrandization on the boundary.

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Dedicated to the memory of
Academician Viktor Pavlovich Maslov

1. INTRODUCTION

Grand Lebesgue spaces in real analysis were introduced and studied in [1] in connection with the applications to PDE's. For generalized grand Lebesgue spaces $L^{p),\theta}(\Omega)$ see [2]–[4]. There followed a boom of studies of grand spaces generated by various spaces of measurable functions. We refer for instance to [5], [6], and references therein. Note also that in the papers [7]–[9] there were studied grand spaces in the setting of an arbitrary Banach function space.

In the case of holomorphic functions, grand Bergman spaces over the unit disk first appeared in [10] (see also [11]). Since holomorphic functions behave well in inner points of the unit disk, it was expected that one needs aggrandization only on the boundary, though the tool to justify this was not clear. The language of local grand spaces developed in real analysis for Lebesgue spaces in [12]–[14] proves to be an appropriate tool for this goal. Since besides Bergman spaces we also aim to consider Bergman–Morrey spaces, we extend the approach of local aggrandization to the case of arbitrary Banach spaces of functions on a metric space.

We hope that this extension may be useful in applications in real and complex analysis.

Using the above-mentioned local aggrandization, in particular for Bergman spaces we prove the following equivalence:

$$\sup_{0<\varepsilon<p-1} \varepsilon^\theta \left(\int_{\mathbb{D}} |f(z)|^{p-\varepsilon}\, \mathrm{dA}(z) \right)^{\frac{1}{p-\varepsilon}} \asymp \sup_{0<\varepsilon<p-1} \varepsilon^\theta \left(\int_{\mathbb{D}} |f(z)|^{p} (1-|z|)^{\lambda\varepsilon}\, \mathrm{dA}(z) \right)^{\frac{1}{p-\varepsilon}} \tag{1.1}$$

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for all $\lambda > 0$ and $l > 0$. In fact, we prove a more general statement with $(1-|z|)^{\lambda\varepsilon}$ replaced by $a(1-|z|)^{\varepsilon}$ under some assumptions on the function a, called an *aggrandizer*.

Moreover, we generalize this to the case of weighted Bergman and Bergman–Morrey spaces. We also pay special attention to weaken assumptions on the function a under which one of the norms in (1.1) just dominates another one.

In these results, estimates of the growth of holomorphic functions near the boundary play an important role. For Bergman–Morrey spaces such estimates are known, see [15], [16]. For weighted Bergman spaces, we provide such estimates in this paper.

The paper is organized as follows. In Sec. 2, we provide some necessary technicalities. Section 3 contains presentation of local aggrandization of Banach spaces over metric spaces. In Sec. 4, we prove our main results.

2. PRELIMINARIES

For a function a positive on $\mathbb{R}_+$, its Matuszewska–Orlicz indices $m(a)$ and $M(a)$ were introduced in [17], see also [18] and [5, Sec. 2.2], where the properties of these indices are given in a form convenient for us. These indices are defined by

$$m(a) = \sup_{0<t<1} \frac{\ln\left(\overline{\lim}_{h\to 0} \frac{a(ht)}{a(h)}\right)}{\ln t},$$

$$M(a) = \sup_{t>1} \frac{\ln\left(\overline{\lim}_{h\to 0} \frac{a(ht)}{a(h)}\right)}{\ln t}.$$

Note that

$$m(t^{\alpha}) = \alpha, \quad m(t^{\alpha}a(t)) = \alpha + m(a), \quad m(a(t)^{\beta}) = \beta m(a), \quad m\left(\frac{1}{a}\right) = -M(a)$$

for $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}_+$.

A positive function a on $(0, \mathcal{D})$ is said to be quasi-monotone if there exist $\alpha, \beta \in \mathbb{R}$ such that $a(t)t^{-\alpha}$ is almost increasing and $a(t)t^{-\beta}$ is almost decreasing. A quasi-monotone function has finite indices, and

$$m(a) = \sup\{\alpha\colon a(t)t^{-\alpha} \text{ is almost increasing}\},$$

$$M(a) = \inf\{\beta\colon a(t)t^{-\beta} \text{ is almost decreasing}\}.$$

Everywhere in the sequel, when considering the indices of a function, we assume it to be quasi-monotone near the origin.

For a function $\varphi : [0,\infty) \to [0,\infty]$, we say that it satisfies the doubling condition if there exists a $C_{(2)} > 0$ such that $\varphi(2t) \le C_{(2)}\varphi(t)$, $t > 0$. A function φ satisfies the reverse doubling condition if there exists a $c_{(2)} > 0$ such that $\varphi(t) \le c_{(2)}\varphi(2t)$, $t > 0$.

3. LOCAL AGGRANDIZATION OF FUNCTION SPACES OVER METRIC SPACES

3.1. Definitions and Main Properties

Let (Λ, d) be a metric space, and let

$$\mathcal{D} = \operatorname{diam}\Lambda, \qquad 0 < \mathcal{D} \le \infty.$$

Given a closed nonempty subset $F \subset \Lambda$, denote

$$\delta_F(x) = \inf_{y\in F} d(x,y), \qquad x \in \Lambda.$$

Let $X = X(\Lambda)$ be an arbitrary normed space of functions $f : \Lambda \to \mathbb{C}$, and let $\|\cdot\|_X$ be its norm.

With the space X we associate the normed space $X_w = X(\Lambda, w)$ depending on a function parameter $w : \Lambda \to \mathbb{R}_+$, and assume that

$$X(\Lambda, w)\big|_{w=1} = X.$$

By $\|\cdot\|_{X,w}$ we denote the norm on X_w. We assume that w is a weight on Λ; i.e., w is a nonnegative function.

Everywhere in the sequel, we assume that the space X_w possesses the following property (lattice property with respect to weights): for any two weights u, v such that $u \le v$, the following inequality holds:

$$\|f\|_{X,u} \le \|f\|_{X,v}. \tag{3.1}$$

Definition 3.1. Let $F \subset \Lambda$ be a closed nonempty set. For a positive almost increasing function $a \in L^\infty(0, \mathcal{D})$, $a(0+) = 0$, we define the local grand space

$$X^g_{F,a,\theta} = X^g_{F,a,\theta}(\Lambda)$$

related to the space X by the norm

$$\|f\|_{X^g_{F,a,\theta}} = \sup_{0<\varepsilon<l} \left(\varepsilon^\theta \|f\|_{X,(a\circ\delta_F)^\varepsilon}\right), \quad l > 0. \tag{3.2}$$

The function a used in the definition of $X^g_{F,a,\theta}$ will be referred to as aggrandizer.

The embedding

$$X \subset X^g_{F,a,\theta} \tag{3.3}$$

holds, because $a \in L^\infty(0, \mathcal{D})$ by definition.

Lemma 3.2. *The space $X^g_{F,a,\theta}$ is independent of the choice of $l > 0$ up to equivalence of norms*:

$$\|f\|_{X^g_{F,a,\theta}}\big|_{l=l_1} \le \|f\|_{X^g_{F,a,\theta}}\big|_{l=l_2} \le C\|f\|_{X^g_{F,a,\theta}}\big|_{l=l_1}, \qquad l_1 < l_2.$$

Proof. Indeed, let $l_1 < l_2$. Then, obviously,

$$\|f\|_{X^g_{F,a,\theta}}\big|_{l=l_1} \le \|f\|_{X^g_{F,a,\theta}}\big|_{l=l_2}.$$

Also, since

$$a(\delta_F(\cdot))^\varepsilon = a(\delta_F(\cdot))^{\varepsilon-l_1} a(\delta_F(\cdot))^{l_1} \le C_0 a(\delta_F(\cdot))^{l_1},$$

it follows that for $l_1 < \varepsilon < l_2$ we have

$$\varepsilon^\theta \le l_1^\theta \left(\frac{l_2}{l_1}\right)^\theta = C_1 l_1^\theta,$$

and

$$\begin{aligned}\|f\|_{X^g_{F,a,\theta}}\Big|_{l=l_2} &= \max\left\{\|f\|_{X^g_{F,a,\theta}}\big|_{l=l_1}, \sup_{l_1<\varepsilon<l_2}\left(\varepsilon^\theta\|f\|_{X,(a\circ\delta_F)^\varepsilon}\right)\right\}\\ &\le \max\left\{\|f\|_{X^g_{F,a,\theta}}\big|_{l=l_1}, C_0C_1l_1^\theta\|f\|_{X,(a\circ\delta_F)^{l_1}}\right\}\\ &\le C_0C_1\|f\|_{X^g_{F,a,\theta}}\big|_{l=l_1}.\end{aligned}$$

This completes the proof. □

In view of Lemma 3.2, we do not introduce the parameter l in the notation of the space $X^g_{F,a,\theta}$.

Lemma 3.3. *Let $s \in (0, \mathcal{D})$ be fixed. The norm* (3.2) *is equivalent to*

$$\sup_{0<\varepsilon<l} \Big(\varepsilon^{\theta} \|\chi_{\{\delta_F(\cdot)<s\}} f\|_{X,(a\circ\delta_F)^{\varepsilon}}\Big) + \|\chi_{\{\delta_F(\cdot)\ge s\}} f\|_X. \tag{3.4}$$

Proof. The proof is straightforward having in mind that $a \in L^{\infty}(0, \mathcal{D})$. □

Theorem 3.4. *The following statements are valid.*

(1) *If there exists a number $\alpha > 0$ such that $a(t) \le Cb(t)^{\alpha}$, $t \in (0, \mathcal{D})$, then*

$$X^g_{F,b,\theta} \hookrightarrow X^g_{F,a,\theta}.$$

(2) *If the function a is almost increasing near the origin and F_1 and F_2 are closed nonempty sets such that $F_1 \subseteq F_2 \subset \Lambda$, then*

$$X^g_{F_1,a,\theta} \hookrightarrow X^g_{F_2,a,\theta}.$$

Proof. To prove the first statement, it suffices to use Lemma 3.2. For the proof of the second statement, just note that $\delta_{F_2}(x) \le \delta_{F_1}(x)$, and then it remains to use the fact that the function a is almost increasing on $(0, \mathcal{D})$. □

Lemma 3.5. *Let $\nu > 0$. Then*

$$X^g_{F,a,\theta} = X^g_{F,a^{\nu},\theta}$$

up to equivalence of norms.

Proof. In fact, the statement of the lemma follows from Theorem 3.4. An alternative straightforward proof is obtained by redenoting $\nu\varepsilon = \varepsilon_1$ and using Lemma 3.2. □

Theorem 3.6. *Let a and b be almost increasing functions on $(0, \mathcal{D})$. If $m(a) > 0$ and $m(b) > 0$, then*

$$X^g_{F,a,\theta} = X^g_{F,b,\theta}$$

up to equivalence of norms.

Proof. In view of Lemma 3.3 and Theorem 3.4, it suffices to refer to the fact that for an arbitrarily small $\eta > 0$ there exist constants $c(\eta)$ and $C(\eta)$ such that

$$c(\eta)t^{M(a)+\eta} \le a(t) \le C(\eta)t^{m(a)-\eta}, \quad t \in (0, \delta), \quad \text{where } \delta \in (0, \mathcal{D}),$$

see [17], [18]. □

3.2. The Embedding $X \subset X^g_{F,a,\theta}$ is Strict in General

We will need an additional assumptions about the space X.

Assume that characteristic functions of balls in Λ belong to X and

$$\|\chi_{B(x_0,r)}\|_X \le \|\chi_{B(x_0,\rho)}\|_X, \qquad r < \rho. \tag{3.5}$$

In the case of constant weights $w = C$, there exists a nondecreasing function $\varphi_X : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|f\|_{X,C} \le \varphi_X(C)\|f\|_X. \tag{3.6}$$

Note that

$$\varphi_X(t) = t^{\frac{1}{p}}$$

for Lebesgue and Morrey spaces if the weight is interpreted as measure (see the definitions of Lebesgue and Morrey spaces in Sec. 4). We show that, in general, the embedding (3.3), i.e., $X \subset X^g_{F,a,\theta}$, is strict. To this end, in view of Theorem 3.4, it suffices to consider the case of $F = \{x_0\}$.

Let us denote

$$\nu_X(\rho) = \|\chi_{B(x_0,\rho)}\|_X.$$

The function ν_X is known as the fundamental function of X.

Theorem 3.7. *Assume that* (3.5) *holds. Let* $x_0 \in \Lambda$. *Assume that the norm of* X *is absolutely continuous. The function*

$$f_0(x) = \nu_X\left(d(x_0,x)\right)^{-1}, \qquad x \in \Lambda \tag{3.7}$$

does not belong to X.

Proof. Let $r \in (0,\mathcal{D})$ be such that $B(x_0,r) \subset \Lambda$. If we assume that $f_0 \in X$, then we have

$$\|f_0\chi_{B(x_0,r)}\|_X = \left\|\frac{\chi_{B(x_0,r)}(x)}{\|\chi_{B(x_0,d(x_0,x))}\|_X}\right\|_X \geq \frac{\|\chi_{B(x_0,r)}\|_X}{\|\chi_{B(x_0,r)}\|_X} = 1.$$

Here we have used the inequality

$$\|\chi_{B(x_0,d(x_0,x))}\|_X \leq \|\chi_{B(x_0,r)}\|_X,$$

which is true for all $x \in \Lambda$ such that $d(x_0,x) \leq r$. This contradicts the absolute continuity property of the norm on X. □

In the next theorem, we provide conditions on the space X under which f_0 belongs to the corresponding grand space $X^g_{F,a,\theta}$.

Theorem 3.8. *Let* $m(a) > 0$. *Assume that* $x_0 \in \Lambda$, $F = \{x_0\}$, *and properties* (3.6) *and* (3.5) *hold. Then the function* f_0 *defined in* (3.7) *belongs to* $X^g_{F,a,\theta}$ *provided that the following condition is satisfied*:

$$\sup_{\varepsilon\in(0,r)}\left\{\varepsilon^\theta\int_0^r \frac{\nu_X(t)}{\nu_X(\frac{t}{2})}\varphi_X(t^\varepsilon)\frac{dt}{t}\right\} < \infty.$$

Proof. We keep in mind that

$$F = \{x_0\}, \qquad \delta_F(x) = d(x_0,x).$$

In view of Theorem 3.6, we can take $a(t) = t$ if Λ is bounded and $a(t) = t$ for $0 < t < 1$ and $a(t) = 1$ for $t \geq 1$ if Λ is unbounded.

Further, we can use the norm $\|f_0\|_{X^g_{F,a,\theta}}$ in the equivalent form (3.4), and so we can replace Λ by a ball $B(x_0,r)$ in Λ; i.e.,

$$\|f_0\|_{X^g_{F,a,\theta}} \simeq \sup_{\varepsilon\in(0,r)}\{\varepsilon^\theta\|\chi_{B(x_0,r)}f_0\|_{X,d(x_0,x)^\varepsilon}\}. \tag{3.8}$$

Let $A_k \subset B(x_0,r)$, $k = 0,1,2\ldots$, stand for the set

$$A_k = \{x \in \Lambda : 2^{-k-1}r < d(x_0,x) < 2^{-k}r\}.$$

For the right-hand side of (3.8), we obtain

$$\begin{aligned}\sup_{\varepsilon\in(0,l)}\{\varepsilon^\theta\|\chi_{B(x_0,r)}f_0\|_{X,d(x_0,x)^\varepsilon}\} &= \sup_{\varepsilon\in(0,l)}\left\{\varepsilon^\theta\left\|\sum_{k=0}^\infty\chi_{A_k}f_0\right\|_{X,d(x_0,x))^\varepsilon}\right\}\\ &\leq \sup_{\varepsilon\in(0,l)}\left\{\varepsilon^\theta\sum_{k=0}^\infty\|\chi_{A_k}f_0\|_{X,d(x_0,x)^\varepsilon}\right\}\end{aligned}$$

$$\leq \sup_{\varepsilon\in(0,l)} \left\{ \varepsilon^\theta \sum_{k=0}^{\infty} \|\chi_{A_k} f_0\|_{X,(2^{-k}r)^\varepsilon} \right\}$$

$$\leq \sup_{\varepsilon\in(0,l)} \left\{ \varepsilon^\theta \sum_{k=0}^{\infty} \varphi_X \left(2^{-k\varepsilon} r^\varepsilon\right) \|\chi_{A_k} f_0\|_X \right\}.$$

Here we consecutively used assumptions (3.1) and (3.6). Recall that

$$f_0(x) = \nu_X(\rho)^{-1}, \qquad \text{where} \qquad \rho = d(x_0, x) \in (0, r).$$

The function ν_X is nondecreasing in $\rho \in (0, r)$. In view of the above estimates, we have, up to some constant $C_1 > 0$,

$$\begin{aligned}
\|f_0\|_{X^g_{F,a,\theta}} &\leq C_1 \sup_{\varepsilon\in(0,l)} \left\{ \varepsilon^\theta \sum_{k=0}^{\infty} \varphi_X \left(2^{-k\varepsilon} r^\varepsilon\right) \|\chi_{A_k} f_0\|_X \right\} \\
&\leq C_1 \sup_{\varepsilon\in(0,l)} \left\{ \varepsilon^\theta \sum_{k=0}^{\infty} \varphi_X \left(2^{-k\varepsilon} r^\varepsilon\right) \nu_X(2^{-k-1}r) \|\chi_{A_k}\|_X \right\} \\
&\leq 2C_1 \sup_{\varepsilon\in(0,l)} \left\{ \varepsilon^\theta \sum_{k=0}^{\infty} \varphi_X \left(2^{-k\varepsilon} r^\varepsilon\right) \nu_X(2^{-k-1}r) \|\chi_{B(x_0,2^{-k}r)}\|_X \right\} \\
&= 2C_1 \sup_{\varepsilon\in(0,l)} \left\{ \varepsilon^\theta \sum_{k=0}^{\infty} \varphi_X \left(2^{-k\varepsilon} r^\varepsilon\right) \nu_X(2^{-k-1}r) \nu_X^{-1}(2^{-k}r) \right\} \\
&= 2C_1 \sup_{\varepsilon\in(0,l)} \left\{ \varepsilon^\theta \sum_{k=0}^{\infty} \varphi_X \left(2^{-k\varepsilon} r^\varepsilon\right) \nu_X(2^{-k-1}r) \nu_X^{-1}(2^{-k}r) \frac{1}{\ln 2} \int_{2^{-k-1}r}^{2^{-k}r} \frac{dt}{t} \right\} \\
&\leq C_1 \frac{1}{\ln 2} \sup_{\varepsilon\in(0,l)} \left\{ \varepsilon^\theta \sum_{k=0}^{\infty} \int_{2^{-k-1}r}^{2^{-k}r} \frac{\nu_X(t)}{\nu_X(\frac{t}{2})} \varphi_X(t^\varepsilon) \frac{dt}{t} \right\} \\
&= 2C_1 \frac{1}{\ln 2} \sup_{\varepsilon\in(0,l)} \left\{ \varepsilon^\theta \int_0^r \frac{\nu_X(t)}{\nu_X(\frac{t}{2})} \varphi_X(t^\varepsilon) \frac{dt}{t} \right\}.
\end{aligned}$$

Here we have used (3.5) and taken into account the inequality $\|\chi_{A_k}\|_X \leq 2\|\chi_{B(x_0,2^{-k}r)}\|_X$. This completes the proof. □

Corollary 3.9. *Assume that $x_0 \in \Lambda$, $F = \{x_0\}$, and properties* (3.6) *and* (3.5) *hold. Assume also that*

$$\int_0^\delta \nu_X(s) \frac{ds}{s} < \infty$$

for some $\delta > 0$ and the function ν_X possesses the doubling property in a right-sided neighbourhood of the origin.

Then the function f_0 defined in (3.7) *belongs to $X^g_{F,a,\theta}$ provided that $\theta \geq 1$.*

4. ON COINCIDENCE OF GRAND SPACES OF HOLOMORPHIC FUNCTIONS WITH THE LOCAL COUNTERPARTS

4.1. Preliminaries on Spaces

Let $\Lambda = \mathbb{D}$ be the unit disk on the complex plane $\mathbb{C}$. We identify $\mathbb{R}^2 \equiv \mathbb{C}$, so that z will stand for $(x, y) = z = x + iy$. Let $\mathrm{dA}(z) = \frac{1}{\pi} dxdy$ be the Lebesgue measure on the unit disk $\mathbb{D}$ normalized so that $|\mathbb{D}| = 1$. The norm on $L^p(\mathbb{D})$ is given by

$$\|f\|_{L^p(\mathbb{D})} = \left(\int_{\mathbb{D}} |f(z)|^p \, \mathrm{dA}(z) \right)^{\frac{1}{p}}, \qquad 1 \leq p < \infty.$$

Let $\rho = \rho(t)$. In the sequel, we always assume that for arbitrary $\delta \in (0,1)$

$$0 < \inf_{t\in(\delta,1)} \rho(t) \le \sup_{t\in(\delta,1)} \rho(t) < \infty.$$

The weighted Lebesgue space $L^p(\mathbb{D},\rho)$ is defined by the norm

$$\|f\|_{L^p(\mathbb{D},\rho)} = \left(\int_{\mathbb{D}} |f(z)|^p \rho(1-|z|)\,\mathrm{dA}(z)\right)^{\frac{1}{p}}, \qquad 1 \le p < \infty.$$

Let $\mathcal{A}^p(\mathbb{D})$ and $\mathcal{A}^p(\mathbb{D},\rho)$ stand for the subspaces of $L^p(\mathbb{D})$ and $L^p(\mathbb{D},\rho)$, respectively, which consist of functions holomorphic in $\mathbb{D}$. They are called the Bergman and weighted Bergman spaces, respectively.

For $1 < p < \infty$ and $\theta > 0$, the grand Lebesgue space $L^{p),\theta}(\mathbb{D})$ consists of all functions f measurable on $\mathbb{D}$ such that

$$\|f\|_{L^{p),\theta}(\mathbb{D})} := \sup_{0<\varepsilon<p-1} \varepsilon^\theta \|f\|_{L^{p-\varepsilon}(\mathbb{D})} < \infty,$$

and the grand weighted Lebesgue space $L^{p),\theta}(\mathbb{D},\rho)$ consists of all functions f measurable on $\mathbb{D}$ such that

$$\|f\|_{L^{p),\theta}(\mathbb{D},\rho)} := \sup_{0<\varepsilon<p-1} \varepsilon^\theta \|f\|_{L^{p-\varepsilon}(\mathbb{D},\rho)} < \infty.$$

The corresponding subspaces of holomorphic functions will be denoted by $\mathcal{A}^{p),\theta}(\mathbb{D})$ and $\mathcal{A}^{p),\theta}(\mathbb{D},\rho)$, respectively.

There arises a natural question as to whether there is a difference between a space of holomorphic functions and the corresponding grand space. The answer is known for $\mathcal{A}^p(\mathbb{D})$ and $\mathcal{A}^{p),\theta}(\mathbb{D})$. More precisely,

$$\frac{1}{(1-z)^\lambda} \in \mathcal{A}^p(\mathbb{D}) \Leftrightarrow \lambda < \frac{2}{p},$$

see [19, p. 78], while it is easily checked that

$$\frac{1}{(1-z)^{\frac{2}{p}}} \in \mathcal{A}^{p),\theta}(\mathbb{D}), \qquad \theta = 1.$$

Moreover, as was proved in [10],

$$g_\theta(z) = \frac{1}{(1-z)^{\frac{2}{p}}} \ln^{\frac{\theta-1}{p}} \frac{e}{1-z} \in \mathcal{A}^{p),\theta}(\mathbb{D})$$

under an appropriate choice of the branch of the logarithmic function.

Similar results can be proved for weighted spaces and Morrey spaces, but we do not touch this question here.

Let $D(z,r)$ denote the Euclidean disk in $\mathbb{C}$ with center z and radius r. Everywhere in the sequel, the function φ is positive and bounded on $(0,1)$.

The Morrey space $L^{p,\varphi}(\mathbb{D})$ is defined as the set of all measurable functions f on $\mathbb{D}$ such that

$$\|f\|_{L^{p,\varphi}(\mathbb{D})} := \sup_{z\in\mathbb{D}, r\in(0,1)} \frac{\varphi(r)}{r^{\frac{2}{p}}} \|f\|_{L^p(D(z,r)\cap\mathbb{D})} < \infty.$$

The grand Morrey space $L^{p),\theta,\varphi}(\mathbb{D})$ is defined as the set of all measurable functions f on $\mathbb{D}$ such that

$$\|f\|_{L^{p),\theta,\varphi}(\mathbb{D})} := \sup_{z\in\mathbb{D}, r\in(0,1)} \frac{\varphi(r)}{r^{\frac{2}{p}}} \|f\|_{L^{p),\theta}(D(z,r)\cap\mathbb{D})} < \infty.$$

4.2. Growth of Holomorphic Functions

We need the following technical lemma.

Lemma 4.1. *Let either ρ be almost increasing and satisfying the doubling condition on $(0,1)$ or ρ be almost decreasing on $(0,1)$ and satisfy $m(\rho) > -1$. Then*

$$\int_{|z-w|<\delta} \rho(1-|w|)\,\mathrm{dA}(w) \le C\rho(\delta)\delta^2.$$

Proof. Let us denote

$$I(\delta, z) = \int_{|w|<\delta} \rho(1-|z-w|)\,\mathrm{dA}(w).$$

We note that

$$|z| - |w| \le |z-w| \le |z| + |w|;$$

hence

$$1 - |z| - |w| \le 1 - |z-w| \le 1 - |z| + |w|.$$

Let ρ be almost increasing and satisfy the doubling condition; then we use $1-|z-w| \le 1-|z|+|w|$ to estimate

$$I(\delta, z) \le C\rho(2\delta) \int_{|w|<\delta} \mathrm{dA}(w) \le C_1\rho(\delta)\delta^2.$$

Let now ρ be almost decreasing and $m(\rho) > -1$; then we use $1-|z|-|w| \le 1-|z-w|$ to obtain

$$\begin{aligned} I(\delta, z) &\le C \int_{|w|<\delta} \rho(\delta - |w|)\,\mathrm{dA}(w) \\ &= 2C \int_0^\delta t\rho(\delta - t)dt \le 2C\delta \int_0^\delta \rho(t)dt \\ &= 2C\delta \int_0^\delta \frac{t\rho(t)}{t} dt \le C_1\rho(\delta)\delta^2. \end{aligned}$$

Here in the last inequality we have used the known fact that $m(a) > 0$ implies the validity of the Zygmund inequality

$$\int_0^\delta \frac{a(t)}{t} dt \le Ca(\delta);$$

see, e.g., [18, Appendix]. □

The classical $\mathcal{A}^p(\mathbb{D})$ result reads

$$|f(z)| \le \frac{\|f\|_{L^p(\mathbb{D})}}{(1-|z|)^{\frac{2}{p}}}, \qquad f \in \mathcal{A}^p(\mathbb{D}), \quad z \in \mathbb{D}, \quad p > 0. \tag{4.1}$$

This result was extended for a variable exponent Morrey space and an Orlicz space in [15, Theorem 5.1]. This result can be slightly improved with respect to the constant in the estimate in the case of a nonvariable Morrey space, as formulated in the following theorem.

Theorem 4.2. *The following statements hold.*

- *Let $f \in \mathcal{A}^p(\mathbb{D}, \rho)$, $1 < p < \infty$. Let $\rho = \rho(t)$ be either almost decreasing and satisfy the reverse doubling condition or almost increasing and satisfy $M(\rho) < p - 1$. Then*

$$|f(z)| \le \frac{\|f\|_{L^p(\mathbb{D},\rho)}}{(1-|z|)^{\frac{2}{p}}\rho(1-|z|)^{\frac{1}{p}}}, \quad z \in \mathbb{D}. \tag{4.2}$$

- *Let* $f \in \mathcal{A}^{p,\varphi}(\mathbb{D})$. *Then*

$$|f(z)| \le \frac{\|f\|_{L^{p,\varphi}(\mathbb{D})}}{\varphi(1-|z|)}, z \in \mathbb{D}.$$

Proof. Fix $z \in \mathbb{D}$. For all $0 \le r < 1-|z|$ we have

$$|f(z)| \le \frac{1}{2\pi}\int_0^{2\pi} |f(z+re^{i\alpha})|d\alpha, \qquad z \in \mathbb{D}.$$

Integration in the variable r over $(0,\delta)$ for a fixed $\delta \le 1-|z|$ with respect to the measure $2rdr$ gives

$$|f(z)| \le \frac{1}{\delta^2}\int_{D(z,\delta)} |f(w)|\,\mathrm{dA}(w), \ \ z \in \mathbb{D}.$$

Applying the Hölder inequality, we have

$$|f(z)| \le \frac{1}{\delta^2}\, \|f\chi_{D(z,\delta)}\|_{L^p(\mathbb{D},\rho)}\, \|\chi_{D(z,\delta)}\|_{L^{p'}(\mathbb{D},\rho^{1-p'})}.$$

Recall that $\rho=\rho(t)$ is either almost decreasing and satisfies the reverse doubling condition or almost increasing and satisfies $M(\rho) < p-1$. Then the function $\rho^{1-p'}$ is either almost increasing and satisfies the doubling condition or almost decreasing and satisfies $m(\rho_0) > -1$, respectively. In both mentioned cases, Lemma 4.1 yields

$$\begin{aligned}\|\chi_{D(z,\delta)}\|^{p'}_{L^{p'}(\mathbb{D},\rho^{1-p'})} &= \int_{|z-w|<\delta} \rho(1-|w|)^{1-p'}\,\mathrm{dA}(w)\\ &\le C\rho(\delta)^{1-p'}\delta^2.\end{aligned}$$

Choosing $\delta = 1-|z|$, we obtain (4.2).

Now, to prove the second statement, one needs to follow the lines of the proof of [15, Theorem 5.1]. The constant can be taken to be one, because in the proof we can use the classical estimate (4.1) instead of using its extension for the variable exponent, which provides the growth estimate up to a constant. □

4.3. The Case of the Space $L^p(\mathbb{D},\rho)$

We take $F = \mathbb{T}$ and $\delta_{\mathbb{T}}(z) = 1-|z|$ for $z \in \mathbb{D}$.

Recall that the aggrandizer $a(t)$ is assumed to be positive for $t \in (0,1)$ and $a(0)=0$.

We find it convenient to use the notation $L^{p)}_{\mathbb{T},a,\theta}(\mathbb{D},\rho)$ for the aggrandized space for $L^p(\mathbb{D},\rho)$. We will use the notation $\mathcal{A}^{p)}_{\mathbb{T},a,\theta}(\mathbb{D},\rho)$ for the subspace of $L^{p)}_{\mathbb{T},a,\theta}(\mathbb{D},\rho)$ which consists of holomorphic functions. We need to write our definition explicitly for the aggrandized space in this case and in particular to clarify the symbol $\|f\|_{L^p(\mathbb{D},\rho),(a\circ\delta_{\mathbb{T}})^\varepsilon}$:

$$\begin{aligned}\|f\|_{L^{p)}_{\mathbb{T},a,\theta}(\mathbb{D},\rho)} &= \sup_{0<\varepsilon<l}\left(\varepsilon^\theta \|f\|_{L^p(\mathbb{D},\rho),(a\circ\delta_{\mathbb{T}})^\varepsilon}\right)\\ &= \sup_{0<\varepsilon<l}\left(\varepsilon^\theta \int_{\mathbb{D}} |f(z)|^p (a\circ\delta_{\mathbb{T}})(z)^\varepsilon \rho(1-|w|)\,\mathrm{dA}(z)\right)^{\frac{1}{p}}, \quad l>0.\end{aligned}$$

When dealing with the space $\mathcal{A}^{p)}_{\mathbb{T},a,\theta}(\mathbb{D},\rho)$ we will assume that $l < p-1$ (we can do this in view of Lemma 3.2).

Theorem 4.3. *The following statements hold true.*

(1) *Let there exist a* $\delta > 0$ *such that*

$$\int_0^1 \frac{\rho(t)}{a(t)^\delta} dt < \infty.$$

Then

$$\mathcal{A}^{p)}_{\mathbb{T},a,\theta}(\mathbb{D},\rho) \hookrightarrow \mathcal{A}^{p),\theta}(\mathbb{D},\rho). \tag{4.3}$$

(2) *Let any of the two following conditions be satisfied*:

(a) *the function* $\rho = \rho(t)$ *is almost decreasing and satisfies the reverse doubling condition, and there exist* $\eta > 0$, $c_a > 0$, *and* $\varepsilon_0 \in (0, p-1)$ *such that*

$$a(t) \le c_a \big(t^{\frac{2}{p}} \rho(t)^{\frac{1}{p-\varepsilon_0}}\big)^\eta, \qquad t \in (0,1). \tag{4.4}$$

(b) *the function* $\rho = \rho(t)$ *is almost increasing,* $M(\rho) < p-1$*, and there exist* $\eta > 0$ *and* $c_a > 0$ *such that*

$$a(t) \le c_a \big(t^{\frac{2}{p}} \rho(t)^{\frac{1}{p}}\big)^\eta, \qquad t \in (0,1).$$

Then

$$\mathcal{A}^{p),\theta}(\mathbb{D},\rho) \hookrightarrow \mathcal{A}^{p)}_{\mathbb{T},a,\theta}(\mathbb{D},\rho). \tag{4.5}$$

Proof. Let us prove (4.3). With $\nu > 0$ such that $\nu p \le \delta$, we have

$$\begin{aligned}
&\left(\int_{\mathbb{D}} |f(w)|^{p-\varepsilon} \rho(1-|w|)\,\mathrm{dA}(w)\right)^{\frac{1}{p-\varepsilon}} \\
&\qquad = \left(\int_{\mathbb{D}} |f(w)|^{p-\varepsilon} a(1-|w|)^{\nu\frac{\varepsilon}{p}(p-\varepsilon)} a(1-|w|)^{\nu\frac{\varepsilon}{p}(\varepsilon-p)} \rho(1-|w|)\,\mathrm{dA}(w)\right)^{\frac{1}{p-\varepsilon}} \\
&\qquad \le \left(\int_{\mathbb{D}} |f(w)|^{p} a(1-|w|)^{\nu\varepsilon} \rho(1-|w|)\,\mathrm{dA}(w)\right)^{\frac{1}{p}} \\
&\qquad\qquad \times \left(\int_{\mathbb{D}} a(1-|w|)^{\nu(\varepsilon-p)} \rho(1-|w|)\,\mathrm{dA}(w)\right)^{\frac{\varepsilon}{p(p-\varepsilon)}} \\
&\qquad \le \left(\int_{\mathbb{D}} |f(w)|^{p} a(1-|w|)^{\nu\varepsilon} \rho(1-|w|)\,\mathrm{dA}(w)\right)^{\frac{1}{p}} \\
&\qquad\qquad \times \left(\int_{\mathbb{D}} a(1-|w|)^{-\nu p} \rho(1-|w|)\,\mathrm{dA}(w)\right)^{\frac{\varepsilon}{p(p-\varepsilon)}} \\
&\qquad \le c_\nu \left(\int_{\mathbb{D}} |f(w)|^{p} a(1-|w|)^{\nu\varepsilon} \rho(1-|w|)\,\mathrm{dA}(w)\right)^{\frac{1}{p}}.
\end{aligned}$$

Here $c_\nu < \infty$ provided that $\nu p \le \delta$. Therefore,

$$\begin{aligned}
\sup_{0<\varepsilon<l} \varepsilon^\theta \|f\|_{L^{p-\varepsilon}(\mathbb{D},\rho)} &\le c_\nu \sup_{0<\varepsilon<l} \varepsilon^\theta \|f\|_{L^p(\mathbb{D},\rho),(a\circ\delta_{\mathbb{T}})^{\nu\varepsilon}} \\
&\le C_0 c_\nu \sup_{0<\varepsilon<l} \varepsilon^\theta \|f\|_{L^p(\mathbb{D},\rho),(a\circ\delta_{\mathbb{T}})^{\varepsilon}}.
\end{aligned}$$

Here the constant C_0 comes from the fact that the aggrandized spaces with the aggrandizers a and a^ν are the same up to equivalence of norms, see Lemma 3.5. This implies (4.3).

Let us prove the second statement. Assume that condition (a) for the second statement takes place. Given arbitrary $\varepsilon_0 \in (0, p-1)$ we assume that $\varepsilon \in (0, \varepsilon_0)$ in the estimates below. We have

$$\begin{aligned}
&\left(\int_{\mathbb{D}} |f(w)|^p a(1-|w|)^{\beta\varepsilon} \rho(1-|w|)\,\mathrm{dA}(w)\right)^{\frac{1}{p}} \\
&\quad = \left(\int_{\mathbb{D}} |f(w)|^{p-\varepsilon} |f(w)|^{\varepsilon} a(1-|w|)^{\beta\varepsilon} \rho(1-|w|)\,\mathrm{dA}(w)\right)^{\frac{1}{p}} \\
&\quad \le \|f\|^{\frac{\varepsilon}{p}}_{L^{p-\varepsilon}(\mathbb{D},\rho)} \left(\int_{\mathbb{D}} |f(w)|^{p-\varepsilon} (1-|w|)^{-\frac{2\varepsilon}{p-\varepsilon}} \rho(1-|w|)^{-\frac{\varepsilon}{p-\varepsilon}} a(1-|w|)^{\beta\varepsilon} \rho(1-|w|)\,\mathrm{dA}(w)\right)^{\frac{1}{p}} \\
&\quad \le \|f\|^{\frac{\varepsilon}{p}}_{L^{p-\varepsilon}(\mathbb{D},\rho)} \left(\int_{\mathbb{D}} |f(w)|^{p-\varepsilon} (1-|w|)^{-\frac{2\varepsilon}{p}} \rho(1-|w|)^{-\frac{\varepsilon}{p-\varepsilon_0}} a(1-|w|)^{\beta\varepsilon} \rho(1-|w|)\,\mathrm{dA}(w)\right)^{\frac{1}{p}} \\
&\quad \le c_a' \|f\|^{\frac{\varepsilon}{p}}_{L^{p-\varepsilon}(\mathbb{D},\rho)} \left(\int_{\mathbb{D}} |f(w)|^{p-\varepsilon} \rho(1-|w|)\,\mathrm{dA}(w)\right)^{\frac{1}{p}} \\
&\quad = c_a' \|f\|^{\frac{\varepsilon}{p}}_{L^{p-\varepsilon}(\mathbb{D},\rho)} \|f\|^{\frac{p-\varepsilon}{p}}_{L^{p-\varepsilon}(\mathbb{D},\rho)} = c_a' \|f\|_{L^{p-\varepsilon}(\mathbb{D},\rho)},
\end{aligned}$$

where $c_a' = \sup_{0<\varepsilon<l} c_a^{\frac{\varepsilon\beta}{p}}$. In the above estimates, we have used the condition (4.4):

$$a(1-|w|)^{\beta} \le c_a^{\beta}\big((1-|w|)^{\frac{2}{p}} \rho(1-|w|)^{\frac{1}{p-\varepsilon_0}}\big)$$

with $\eta = \frac{1}{\beta}$. If condition (b) of the second statement takes place, then we repeat the above calculations and estimate

$$\rho(1-|w|)^{\frac{1}{p-\varepsilon}} \le \rho(1-|w|)^{\frac{1}{p}}.$$

The remaining part of the argument is the same except that the exponent $\frac{1}{p}$ is used instead of $\frac{1}{p-\varepsilon_0}$.

We recall that the norm in our aggrandized spaces and in the classical grand spaces does not depend on the range of ε up to equivalence. Therefore, the above arguments yield

$$\begin{aligned}
\sup_{0<\varepsilon<l} \varepsilon^{\theta} \|f\|_{L^p(\mathbb{D},\rho),(a\circ\delta_{\mathbb{T}})^{\varepsilon}} &\le C_0 \sup_{0<\varepsilon<l} \varepsilon^{\theta} \|f\|_{L^p(\mathbb{D},\rho),(a\circ\delta_{\mathbb{T}})^{\beta\varepsilon}} \\
&\le C_0 c_a' \sup_{0<\varepsilon<l} \varepsilon^{\theta} \|f\|_{L^{p-\varepsilon}(\mathbb{D},\rho)}.
\end{aligned}$$

Here the constant C_0 comes from the fact that the aggrandized spaces with the aggrandizers a and a^{β} are the same up to equivalence of norms, see Lemma 3.5. This implies the embedding (4.5). ☐

The following corollary is immediate.

Corollary 4.4. *Let assumptions in both parts* (1) *and* (2) *of Theorem* 4.3 *be satisfied. Then the space* $\mathcal{A}^{p),\theta}(\mathbb{D},\rho)$ *coincides with the space* $\mathcal{A}^{p)}_{\mathbb{T},a,\theta}(\mathbb{D},\rho)$, *up to equivalence of norms.*

The case of a power weight is of particular interest.

Theorem 4.5. *Assume that* $\rho(t) = t^{\gamma}$, $-1 < \gamma < p-1$, *there exists a* $\delta > 0$ *such that*

$$\int_0^1 \frac{t^{\gamma}\,dt}{a(t)^{\delta}} < \infty,$$

and there exist $C > 0$ *and* $\eta > 0$ *such that*

$$a(t) \le Ct^{\eta}.$$

Then the space $\mathcal{A}^{p),\theta}(\mathbb{D},\rho)$ *coincides with the space* $\mathcal{A}^{p)}_{\mathbb{T},a,\theta}(\mathbb{D},\rho)$, *up to equivalence of norms.*

Proof. The proof follows by noticing that in the case of power weight $\rho(t) = t^\gamma$, $-1 < \gamma < p-1$, the conditions stated in parts (1) and (2) of Theorem 4.3 are reduced to the only one condition $a(t) \le Ct^\eta$. Certainly, such η may be different in (1) and (2), but then we use Lemma 3.5, which allows us to raise the aggrandizer to any positive power and still obtain the same aggrandized space up to equivalence of norms. Such a trick was demonstrated in the proof of Theorem 4.3. □

We single out a particular but important corollary for the unweighed case: $\rho = 1$. We use the symbols $\mathcal{A}^{p),\theta}(\mathbb{D})$ and $\mathcal{A}^{p)}_{\mathbb{T},a,\theta}(\mathbb{D})$ for the unweighed spaces.

Theorem 4.6. *The space $\mathcal{A}^{p),\theta}(\mathbb{D})$ coincides with the space $\mathcal{A}^{p)}_{\mathbb{T},a,\theta}(\mathbb{D})$ with any aggrandizer a satisfying $m(a) > 0$ up to equivalence of norms.*

4.4. The Case of the Morrey Space $L^{p,\varphi}(\mathbb{D})$

Recall that $F = \mathbb{T}$ and $\delta_{\mathbb{T}}(z) = 1 - |z|$ for $z \in \mathbb{D}$. We find it convenient to use the notation $L^{p),\varphi}_{\mathbb{T},a,\theta}(\mathbb{D})$ for the aggrandized space for $L^{p,\varphi}(\mathbb{D})$. We will use $\mathcal{A}^{p),\varphi}_{\mathbb{T},a,\theta}(\mathbb{D})$ for the subspace of $L^{p),\varphi}_{\mathbb{T},a,\theta}(\mathbb{D})$ which consists of holomorphic functions. Below we explicitly write down the definition of the norm in the aggrandized Morrey space:

$$\|f\|_{L^{p),\varphi}_{\mathbb{T},a,\theta}(\mathbb{D})} = \sup_{0<\varepsilon<l} \left(\varepsilon^\theta \|f\|_{L^{p,\varphi}(\mathbb{D}),(a\circ\delta_{\mathbb{T}})^\varepsilon}\right)$$
$$\equiv \sup_{0<\varepsilon<l} \varepsilon^\theta \sup_{z\in\mathbb{D},r\in(0,1)} \frac{\varphi(r)}{r^{\frac{2}{p}}} \left(\int_{D(z,r)\cap\mathbb{D}} |f(w)|^p (a\circ\delta_{\mathbb{T}})(w)^\varepsilon \,\mathrm{dA}(w)\right)^{\frac{1}{p}}, \quad l > 0.$$

Here we again assume that $l < p-1$.

Theorem 4.7. *The following statements hold true.*

(1) *Let there exist a $\delta > 0$ such that*
$$\int_0^1 \frac{dt}{a(t)^\delta} < \infty.$$
Then $\mathcal{A}^{p),\varphi}_{\mathbb{T},a,\theta}(\mathbb{D}) \hookrightarrow \mathcal{A}^{p),\theta,\varphi}(\mathbb{D})$.

(2) *Let there exist $\beta > 0$ and $c_{a,\varphi} > 0$ such that $a(t)^\beta \le c_{a,\varphi}\varphi(t)$. Then*
$$\mathcal{A}^{p),\theta,\varphi}(\mathbb{D}) \hookrightarrow \mathcal{A}^{p),\varphi}_{\mathbb{T},a,\theta}(\mathbb{D}). \tag{4.6}$$

Proof. Let us prove that $\mathcal{A}^{p),\varphi}_{\mathbb{T},a,\theta}(\mathbb{D}) \hookrightarrow \mathcal{A}^{p),\theta,\varphi}(\mathbb{D})$. By analogy with the proof of Theorem 4.3, let us take $\nu > 0$ such that $\nu p = \delta$, and calculate

$$\left(\int_{D(z,r)\cap\mathbb{D}} |f(w)|^{p-\varepsilon} \,\mathrm{dA}(w)\right)^{\frac{1}{p-\varepsilon}}$$
$$= \left(\int_{D(z,r)\cap\mathbb{D}} |f(w)|^{p-\varepsilon} a(1-|w|)^{\nu\frac{\varepsilon}{p}(p-\varepsilon)} a(1-|w|)^{\nu\frac{\varepsilon}{p}(\varepsilon-p)} \,\mathrm{dA}(w)\right)^{\frac{1}{p-\varepsilon}}$$
$$\le \left(\int_{D(z,r)\cap\mathbb{D}} |f(w)|^p a(1-|w|)^{\nu\varepsilon} \,\mathrm{dA}(w)\right)^{\frac{1}{p}} \left(\int_{D(z,r)\cap\mathbb{D}} a(1-|w|)^{\nu(\varepsilon-p)} \,\mathrm{dA}(w)\right)^{\frac{\varepsilon}{p(p-\varepsilon)}}$$
$$\le \left(\int_{D(z,r)\cap\mathbb{D}} |f(w)|^p a(1-|w|)^{\nu\varepsilon} \,\mathrm{dA}(w)\right)^{\frac{1}{p}} \left(\int_{\mathbb{D}} a(1-|w|)^{\nu(\varepsilon-p)} \,\mathrm{dA}(w)\right)^{\frac{\varepsilon}{p(p-\varepsilon)}}$$

$$\le \left(\int_{D(z,r)\cap\mathbb{D}} |f(w)|^p a(1-|w|)^{\nu\varepsilon}\,\mathrm{dA}(w)\right)^{\frac{1}{p}} \left(\int_{\mathbb{D}} a(1-|w|)^{-\nu p}\,\mathrm{dA}(w)\right)^{\frac{\varepsilon}{p(p-\varepsilon)}}$$
$$\le c_\nu \left(\int_{D(z,r)\cap\mathbb{D}} |f(w)|^p a(1-|w|)^{\nu\varepsilon}\,\mathrm{dA}(w)\right)^{\frac{1}{p}}.$$

Here $c_\nu < \infty$. Hence $\|f\|_{L^{p),\theta,\varphi}(\mathbb{D})} \le c_\nu \|f\|_{L^{p),\varphi}_{\mathbb{T},a^\nu,\theta}(\mathbb{D})} \le C_0 c_\nu \|f\|_{L^{p),\varphi}_{\mathbb{T},a,\theta}(\mathbb{D})}$. Here again we have used Lemma 3.5 for the last inequality above.

Let us prove that $\mathcal{A}^{p),\theta,\varphi}(\mathbb{D}) \hookrightarrow \mathcal{A}^{p),\varphi}_{\mathbb{T},a,\theta}(\mathbb{D})$. We have

$$\left(\int_{D(z,r)\cap\mathbb{D}} |f(w)|^p a(1-|w|)^{\beta\varepsilon}\,\mathrm{dA}(w)\right)^{\frac{1}{p}}$$
$$= \left(\int_{D(z,r)\cap\mathbb{D}} |f(w)|^{p-\varepsilon} |f(w)|^{\varepsilon} (1-|w|)^{\beta\varepsilon}\,\mathrm{dA}(w)\right)^{\frac{1}{p}}$$
$$\le \|f\|^{\frac{\varepsilon}{p}}_{L^{p-\varepsilon,\varphi}(\mathbb{D})} \left(\int_{D(z,r)\cap\mathbb{D}} |f(w)|^{p-\varepsilon} \varphi(1-|w|)^{-\varepsilon} a(1-|w|)^{\beta\varepsilon}\,\mathrm{dA}(w)\right)^{\frac{1}{p}}$$
$$\le c'_{a,\varphi} \|f\|^{\frac{\varepsilon}{p}}_{L^{p-\varepsilon,\varphi}(\mathbb{D})} \left(\int_{D(z,r)\cap\mathbb{D}} |f(w)|^{p-\varepsilon}\,\mathrm{dA}(w)\right)^{\frac{1}{p}}$$
$$= c'_{a,\varphi} \|f\|^{\frac{\varepsilon}{p}}_{L^{p-\varepsilon,\varphi}(\mathbb{D})} \|f\|^{\frac{p-\varepsilon}{p}}_{L^{p-\varepsilon}(D(z,r)\cap\mathbb{D})},$$

where $c'_{a,\varphi} = \sup_{0<\varepsilon<l} c^{\frac{\varepsilon}{p}}_{a,\varphi}$. Hence

$$\sup_{z\in\mathbb{D},r\in(0,1)} \frac{\varphi(r)}{r^{\frac{2}{p}}} \left(\int_{D(z,r)\cap\mathbb{D}} |f(w)|^p a(1-|w|)^{\beta\varepsilon}\,\mathrm{dA}(w)\right)^{\frac{1}{p}}$$
$$\le c'_{a,\varphi} \|f\|^{\frac{\varepsilon}{p}}_{L^{p-\varepsilon,\varphi}(\mathbb{D})} \sup_{z\in\mathbb{D},r\in(0,1)} \frac{\varphi(r)}{r^{\frac{2}{p}}} \|f\|^{\frac{p-\varepsilon}{p}}_{L^{p-\varepsilon}(D(z,r)\cap\mathbb{D})}$$
$$\le c'_{a,\varphi} \|f\|^{\frac{\varepsilon}{p}}_{L^{p-\varepsilon,\varphi}(\mathbb{D})} \sup_{z\in\mathbb{D},r\in(0,1)} \frac{C_0 \varphi(r)^{\frac{p-\varepsilon}{p}}}{\left(r^{\frac{2}{p-\varepsilon}}\right)^{\frac{p-\varepsilon}{p}}} \|f\|^{\frac{p-\varepsilon}{p}}_{L^{p-\varepsilon}(D(z,r)\cap\mathbb{D})}$$
$$= c'_{a,\varphi} C_0 \|f\|^{\frac{\varepsilon}{p}}_{L^{p-\varepsilon,\varphi}(\mathbb{D})} \left(\sup_{z\in\mathbb{D},r\in(0,1)} \frac{\varphi(r)}{r^{\frac{2}{p-\varepsilon}}} \|f\|_{L^{p-\varepsilon}(D(z,r)\cap\mathbb{D})}\right)^{\frac{p-\varepsilon}{p}}$$
$$= c'_{a,\varphi} C_0 \|f\|^{\frac{\varepsilon}{p}}_{L^{p-\varepsilon,\varphi}(\mathbb{D})} \|f\|^{\frac{p-\varepsilon}{p}}_{L^{p-\varepsilon,\varphi}(\mathbb{D})} = c'_{a,\varphi} C_0 \|f\|_{L^{p-\varepsilon,\varphi}(\mathbb{D})},$$

where C_0 is a constant such that $\varphi(r) \le C_0 \varphi(r)^{\frac{p-\varepsilon}{p}}$, $r \in (0,1)$. Therefore, from the above calculation we obtain

$$\|f\|_{L^{p),\varphi}_{\mathbb{T},a,\theta}(\mathbb{D})} \le C_0 \|f\|_{L^{p),\varphi}_{\mathbb{T},a^\beta,\theta}(\mathbb{D})} \le c'_{a,\varphi} C_0 \|f\|_{L^{p),\theta,\varphi}(\mathbb{D})}.$$

In the first inequality above, we have again used the fact that the aggrandized spaces with the aggrandizers a and a^β are the same up to equivalence of norms, see Lemma 3.5. This implies (4.6). □

Corollary 4.8. *Let the aggrandizer a be such that the assumptions in both parts* (1) *and* (2) *of Theorem* 4.7 *are satisfied. Then the space $\mathcal{A}^{p),\theta,\varphi}(\mathbb{D})$ coincides with the space $\mathcal{A}^{p),\varphi}_{\mathbb{T},a,\theta}(\mathbb{D})$, up to equivalence of norms.*

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DATA AVAILABILITY

The authors confirm that all data generated or analyzed during this study are included in this article.

CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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